On L^p Estimates in Homogenization of Elliptic Equations of Maxwell's Type

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Abstract

For a family of second-order elliptic systems of Maxwell's type with rapidly oscillating periodic coefficients in a $C^{1,\alpha}$ domain Ω, we establish uniform estimates of solutions u_ε and $\nabla \times u_\varepsilon$ in $L^p(\Omega)$ for $1 < p \le \infty$. The proof relies on the uniform $W^{1,p}$ and Lipschitz estimates for solutions of scalar elliptic equations with periodic coefficients.

Keywords: Homogenization; Maxwell's Equations; L^p Estimates.

1 Introduction

Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^3$ for some $\alpha > 0$ and n denote the outward unit normal to $\partial\Omega$. Let $A(y) = (a_{ij}(y))$ and $B(y) = (b_{ij}(y))$ be two 3×3 matrices with real entries satisfying the ellipticity conditions:

$$\mu|\xi|^2 \le a_{ij}(y)\xi_i\xi_j \le \frac{1}{\mu}|\xi|^2, \quad \mu|\xi|^2 \le b_{ij}(y)\xi_i\xi_j \le \frac{1}{\mu}|\xi|^2 \tag{1.1}$$

for any $\xi, y \in \mathbb{R}^3$ and some $\mu > 0$. Consider the second-order elliptic system of Maxwell's type:

$$\nabla \times \big(A(x/\varepsilon)\nabla \times u_\varepsilon\big) + B(x/\varepsilon)u_\varepsilon = F + \nabla \times G \quad \text{in } \Omega, \tag{1.2}$$

where u_ε is a vector field in Ω and $\varepsilon > 0$ a small parameter. Given $F, G \in L^2(\Omega; \mathbb{R}^3)$, it follows readily from the Lax-Milgram Theorem that the elliptic system (1.2) has a unique (weak) solution in

$$V_0^2(\Omega) = \big\{u \in L^2(\Omega; \mathbb{R}^3) : \ \nabla \times u \in L^2(\Omega; \mathbb{R}^3) \text{ and } n \times u = 0 \text{ on } \partial\Omega\big\}. \tag{1.3}$$

Moreover, the solution satisfies the estimate

$$\|u_\varepsilon\|_{L^2(\Omega)} + \|\nabla \times u_\varepsilon\|_{L^2(\Omega)} \le C\left\{\|F\|_{L^2(\Omega)} + \|G\|_{L^2(\Omega)}\right\}, \tag{1.4}$$

where C depends only on μ and Ω. Suppose, in addition, that the matrices $A(y)$ and $B(y)$ are periodic with respect to $\mathbb{Z}^3$:

$$A(y + z) = A(y) \quad \text{and} \quad B(y + z) = B(y) \quad \text{for any } y \in \mathbb{R}^3, z \in \mathbb{Z}^3. \tag{1.5}$$

It follows from the theory of homogenization that $u_\varepsilon \to u_0$ weakly in $V_0^2(\Omega)$ as $\varepsilon \to 0$, and u_0 is the unique solution in $V_0^2(\Omega)$ of the homogenized system:

$$\nabla \times \big(A_0 \nabla \times u_0\big) + B_0 u_0 = F + \nabla \times G \quad \text{ in } \Omega, \tag{1.6}$$

where A_0 and B_0 are constant (effective) matrices given by

$$A_0 = \big(\mathcal{H}(A^{-1})\big)^{-1} \quad \text{ and } \quad B_0 = \mathcal{H}(B).$$

We refer the reader to [3, pp.81-91] for the definition of A_0 and B_0 as well as the homogenization theory for (1.2).

In this paper we consider the boundary value problem for the elliptic system (1.2):

$$\begin{cases} \nabla \times \big(A(x/\varepsilon)\nabla \times u_\varepsilon\big) + B(x/\varepsilon)u_\varepsilon = F + \nabla \times G & \text{ in } \Omega, \\ n \times u_\varepsilon = f & \text{ on } \partial\Omega. \end{cases} \tag{1.7}$$

We shall be interested in the estimates of u_ε and $\nabla \times u_\varepsilon$, which are uniform in $\varepsilon > 0$, in $L^p(\Omega)$ for $1 < p \le \infty$, under the ellipticity and periodicity conditions on A and B.

For $1 < p < \infty$, let

$$V^p(\Omega) = \big\{u \in L^p(\Omega; \mathbb{R}^3) : \nabla \times u \in L^p(\Omega; \mathbb{R}^3)\big\}. \tag{1.8}$$

If $f \in L^p(\partial\Omega; \mathbb{R}^3)$ and $n \cdot f = 0$ on $\partial\Omega$, we will use $\text{Div}(f)$ to denote the surface divergence of f on $\partial\Omega$, defined by

$$< \text{Div}(f), \psi >_{W^{-1,p}(\partial\Omega)\times W^{1,p'}(\partial\Omega)} = -\int_{\partial\Omega} < f, \nabla_{\tan}\psi > \, d\sigma, \tag{1.9}$$

where $\psi \in C^\infty(\mathbb{R}^d)$ and $\nabla_{\tan}\psi = \nabla\psi - <\nabla\psi, n> n$ denotes the tangential gradient of ψ on $\partial\Omega$. The following are the main results of the paper.

Theorem 1.1. *Let $1 < p < \infty$ and Ω be a bounded, simply connected, $C^{1,\alpha}$ domain in $\mathbb{R}^3$ with connected boundary. Suppose that A and B satisfy conditions (1.1) and (1.5) and that A, B are Hölder continuous. Let $F, G \in L^p(\Omega; \mathbb{R}^3)$ and $f \in L^p(\partial\Omega; \mathbb{R}^3)$ with $n \cdot f = 0$ on $\partial\Omega$ and $\text{Div}(f) \in W^{-\frac{1}{p},p}(\partial\Omega)$. Then the boundary value problem (1.7) has a unique solution in $V^p(\Omega)$. Moreover, the solution u_ε satisfies*

$$\begin{aligned} &\|u_\varepsilon\|_{L^p(\Omega)} + \|\nabla \times u_\varepsilon\|_{L^p(\Omega)} \\ &\qquad \le C_p \left\{ \|F\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)} + \|f\|_{L^p(\partial\Omega)} + \|\text{Div}(f)\|_{W^{-\frac{1}{p},p}(\partial\Omega)} \right\}, \end{aligned} \tag{1.10}$$

where the constant C_p is independent of $\varepsilon > 0$.

Theorem 1.2. *Let Ω be a bounded, simply connected, $C^{1,\alpha}$ domain in $\mathbb{R}^3$ with connected boundary. Suppose that A and B satisfy conditions (1.1) and (1.5) and that A, B are Hölder continuous. Also assume that A is symmetric. Let $F, G \in C^\gamma(\Omega; \mathbb{R}^3)$ and $f \in C^\gamma(\partial\Omega; \mathbb{R}^3)$ with $n \cdot f = 0$ on $\partial\Omega$ and $\text{Div}(f) \in C^\gamma(\partial\Omega)$ for some $\gamma > 0$. Let u_ε be the unique solution of (1.7) in $V^2(\Omega)$. Then $u_\varepsilon, \nabla \times u_\varepsilon \in L^\infty(\Omega; \mathbb{R}^3)$, and*

$$\begin{aligned} &\|u_\varepsilon\|_{L^\infty(\Omega)} + \|\nabla \times u_\varepsilon\|_{L^\infty(\Omega)} \\ &\qquad \le C_\gamma \left\{ \|F\|_{C^\gamma(\Omega)} + \|G\|_{C^\gamma(\Omega)} + \|f\|_{C^\gamma(\partial\Omega)} + \|\text{Div}(f)\|_{C^\gamma(\partial\Omega)} \right\}, \end{aligned} \tag{1.11}$$

where the constant C_γ is independent of $\varepsilon > 0$.

Besides the interest in their own rights, uniform regularity estimates are an important tool in the study of convergence problems for solutions u_ε, eigenfunctions, and eigenvalues in the theory of homogenization. For the elliptic systems

$$-\operatorname{div}\big(A(x/\varepsilon)\nabla u_\varepsilon\big) = F \quad \text{ in } \Omega, \tag{1.12}$$

where $A(y) = \big(a_{ij}^{\alpha\beta}(y)\big)$ with $1 \le i, j \le d$ and $1 \le \alpha, \beta \le m$ is uniform elliptic, periodic, and Hölder continuous, uniform $W^{1,p}$ estimates, Hölder estimates, and Lipschitz estimates were established in [1] [2] for solutions in $C^{1,\alpha}$ domains with the Dirichlet boundary condition. Analogous results for solutions in $C^{1,\alpha}$ domains with the Neumann boundary conditions were recently obtained in [8]. We mention that for suitable solutions of $\operatorname{div}\big(A(x/\varepsilon)\nabla u_\varepsilon\big) = 0$ in a Lipschitz domain Ω, under the additional symmetry condition $a_{ij}^{\alpha\beta}(y) = a_{ji}^{\beta\alpha}(y)$, the following uniform L^2 Rellich estimates:

$$\left\|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\right\|_{L^2(\partial\Omega)} \approx \|\nabla_{\tan} u_\varepsilon\|_{L^2(\partial\Omega)} \tag{1.13}$$

were proved in [9] [10], where $\partial u_\varepsilon/\partial\nu_\varepsilon$ and $\nabla_{\tan} u_\varepsilon$ denote the conormal derivative and tangential gradient of u_ε on $\partial\Omega$, respectively. The proof for the Lipschitz estimates in [8] relies on the L^2 Relllich estimates in [10]. As a result, the Lipschitz estimates in [8] for solutions with the Neumann boundary conditions, which are used in the proof of Theorem 1.2, were established under the additional symmetry condition.

To prove Theorems 1.1 and 1.2, our basic idea is to reduce the study of (1.2) to that of a scalar uniform elliptic equation of divergence form. This uses the well-known fact that on a simply connected domain Ω in $\mathbb{R}^3$, $u \in L^2(\Omega; \mathbb{R}^3)$ and $\nabla \times u = 0$ in Ω imply that $u = \nabla P$ in Ω for some scalar function $P \in H^1(\Omega)$. It also relies on the fact that on a bounded C^1 domain Ω with connected boundary, $u \in L^p(\Omega; \mathbb{R}^3)$ and $\operatorname{div}(u) = 0$ in Ω imply that $u = \nabla \times v$ in Ω for some $v \in W^{1,p}(\Omega; \mathbb{R}^3)$. The approach allows us to reduce the estimates (1.10) and (1.11) to the $W^{1,p}$ and Lipschitz estimates for solutions of the scalar elliptic equation

$$-\operatorname{div}\big(A(x/\varepsilon)(\nabla w_\varepsilon + g)\big) = F \quad \text{ in } \Omega. \tag{1.14}$$

We point out that both the Dirichlet condition and the Neumann condition for the elliptic equation (1.14) are needed to handle the system (1.2).

The rest of the paper is organized as follows. In Section 2 we collect some basic facts related to the divergence and curl operators, which will be needed in Section 4. In Section 3 we establish the $W^{1,p}$ and Lipschitz estimates for (1.14) in a bounded $C^{1,\alpha}$ domain. While the $W^{1,p}$ estimates for (1.14) follow readily from those for (1.12) with $m = 1$ in [1, 2] and [8], the desired Lipschitz estimates require some additional argument, involving the Green and Neumann functions for (1.12). The proof of Theorem 1.1 is given in Section 4, and the proof of Theorem 1.2 in Section 5. Finally, we point out that under the additional assumption that A is Lipschitz continuous, B is a constant matrix, and Ω is $C^{1,1}$, it follows from the estimate (1.10) that

$$\|u_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \left\{\|F\|_{L^p(\Omega)} + \|\operatorname{div}(F)\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)} + \|f\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\right\} \tag{1.15}$$

for $1 < p < \infty$ (see Remark 2.4).

2 Some preliminaries

The materials in this section are more or less known.

Theorem 2.1. *Let Ω be a bounded, simply-connected, Lipschitz domain in $\mathbb{R}^3$. Suppose that $u \in L^p(\Omega;\mathbb{R}^3)$ for some $1 < p < \infty$ and $\nabla \times u = 0$ in Ω. Then $u = \nabla P$ in Ω for some $P \in W^{1,p}(\Omega)$.*

Proof. The case $p > 2$ follows directly from the case $p = 2$, which is well known. The case $p < 2$ may be proved in the same manner as in the case $p = 2$ (see e.g. [7, pp.31-32]). $\square$

We will use $W^{t,p}(\partial\Omega)$ to denote the Sobolev-Besov space of order t and exponent p on $\partial\Omega$ for $-1 < t < 1$ and $1 < p < \infty$. Note that the dual of $W^{t,p}(\partial\Omega)$ is given by $W^{-t,q}(\partial\Omega)$, where $q = p' = \frac{p}{p-1}$.

Theorem 2.2. *Let Ω be a bounded C^1 domain in $\mathbb{R}^3$ with connected boundary. Let $g \in L^p(\Omega;\mathbb{R}^3)$ for some $1 < p < \infty$. Suppose that $\mathrm{div}(g) = 0$ in Ω. Then there exists $h \in W^{1,p}(\Omega;\mathbb{R}^3)$ such that $\nabla \times h = g$ in Ω. Moreover, $\mathrm{div}(h) = 0$ in Ω and $\|h\|_{W^{1,p}(\Omega)} \leq C_p \|g\|_{L^p(\Omega)}$, where C_p depends only on p and Ω.*

Proof. The result is well known for smooth domains. The proof for the case of C^1 domains is similar. We provide a proof, which follows the lines in [7] and [4], for the sake of completeness.

We first note that if $u \in L^p(\mathbb{R}^3;\mathbb{R}^3)$ with $\mathrm{supp}(u) \subset B = B(0,R)$ and $\mathrm{div}(u) = 0$ in $\mathbb{R}^3$, then there exists $v \in W^{1,p}(B;\mathbb{R}^3)$ such that $\nabla \times v = u$ in B, $\mathrm{div}(v) = 0$ in B, and $\|v\|_{W^{1,p}(B)} \leq C_p \|u\|_{L^p(\mathbb{R}^3)}$. To see this, we let $v = \nabla \times w$, where

$$w(x) = \int_{\mathbb{R}^3} \Gamma(x-y)u(y)\,dy,$$

and $\Gamma(x) = (4\pi|x|)^{-1}$ is the fundamental solution for $-\Delta$ in $\mathbb{R}^3$, with pole at the origin. It follows from $\mathrm{div}(u) = 0$ in $\mathbb{R}^3$ that $\mathrm{div}(w) = 0$ in $\mathbb{R}^3$. Hence,

$$\nabla \times v = \nabla \times (\nabla \times w) = -\Delta w + \nabla(\mathrm{div}(w)) = -\Delta w = u.$$

Clearly, $\mathrm{div}(v) = 0$ in $\mathbb{R}^3$. Also, by the Calderón-Zygmund estimate and fractional intergal estimate,

$$\|v\|_{W^{1,p}(B)} \leq C\|\nabla w\|_{W^{1,p}(B)} \leq C_p \|u\|_{L^p(\mathbb{R}^3)},$$

where C_p may depend on R.

We now consider the case where Ω is a bounded C^1 domain with connected boundary. Choose a ball $B = B(0,R)$ such that $\overline{\Omega} \subset B(0,R/4)$. Since $\partial\Omega$ is connected, $B \setminus \overline{\Omega}$ is a bounded (connected) C^1 domain. Also, $g \in L^p(\Omega;\mathbb{R}^3)$ and $\mathrm{div}(g) = 0$ in Ω imply that $n \cdot g \in W^{-\frac{1}{p},p}(\partial\Omega)$ and

$$< n \cdot g, 1 >_{W^{-\frac{1}{p},p}(\partial\Omega) \times W^{\frac{1}{p},p'}(\partial\Omega)} = 0.$$

It follows from [5] that there exists $f \in W^{1,p}(B \setminus \overline{\Omega})$ such that $\Delta f = 0$ in $B \setminus \overline{\Omega}$, $\frac{\partial f}{\partial n} = n \cdot g$ on $\partial\Omega$, and $\frac{\partial f}{\partial n} = 0$ on ∂B. Moreover,

$$\|\nabla f\|_{L^p(B\setminus\overline{\Omega})} \leq C_p \|n \cdot g\|_{W^{-\frac{1}{p},p}(\partial\Omega)} \leq C_p \|g\|_{L^p(\Omega)}. \tag{2.1}$$

Define

$$\widetilde{g} = \begin{cases} g & \text{in } \Omega, \\ \nabla f & \text{in } B \setminus \overline{\Omega}, \\ 0 & \text{in } \mathbb{R}^3 \setminus B. \end{cases}$$

Note that $\widetilde{g} \in L^p(\mathbb{R}^3; \mathbb{R}^3)$ and for any $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} \int_{\mathbb{R}^3} \widetilde{g} \cdot \nabla \psi \, dx &= \int_\Omega g \cdot \nabla \psi \, dx + \int_{B \setminus \overline{\Omega}} \nabla f \cdot \nabla \psi \, dx \\ &= < n \cdot g, \psi >_{W^{-\frac{1}{p},p}(\partial\Omega) \times W^{\frac{1}{p},p'}(\partial\Omega)} + \int_{B \setminus \overline{\Omega}} \nabla f \cdot \nabla \psi \, dx \\ &= 0. \end{aligned}$$

Thus, $\mathrm{div}(\widetilde{g}) = 0$ in $\mathbb{R}^3$. It follows from the first part of the proof that $\widetilde{g} = \nabla \times h$ in B for some $h \in W^{1,p}(B; \mathbb{R}^3)$ with $\mathrm{div}(h) = 0$ in B. Furthermore,

$$\|h\|_{W^{1,p}(\Omega)} \leq \|h\|_{W^{1,p}(B)} \leq C_p \|\widetilde{g}\|_{L^p(B)} \leq C_p \|g\|_{L^p(\Omega)},$$

where we have used (2.1) for the last inequality. This competes the proof. $\square$

Theorem 2.3. *Let $1 < p < \infty$ and Ω be a bounded, simply-connected, $C^{1,1}$ domain in $\mathbb{R}^3$ with connected boundary. Let $A = A(x)$ be a 3×3 matrix in $\mathbb{R}^3$ satisfying the ellipticity condition (3.1). Also assume that A is Lipschitz continuous. Then, for any $u \in L^p(\Omega; \mathbb{R}^3)$ such that the right hand side of (2.2) is finite,*

$$\|\nabla u\|_{L^p(\Omega)} \leq C_p \Big\{ \|\nabla \times u\|_{L^p(\Omega)} + \|\mathrm{div}(Au)\|_{L^p(\Omega)} + \|n \times u\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \Big\}, \tag{2.2}$$

where C_p depends only on p, Ω, and A.

Proof. Let u be a function in $L^p(\Omega; \mathbb{R}^3)$ such that the right hand side of (2.2) is finite. Let $g = \nabla \times u$ in Ω. Then $g \in L^p(\Omega; \mathbb{R}^3)$ and $\mathrm{div}(g) = 0$ in Ω. In view of Theorem 2.2, there exists $h \in W^{1,p}(\Omega; \mathbb{R}^3)$ such that $g = \nabla \times h$ in Ω, $\mathrm{div}(h) = 0$ in Ω, and

$$\|h\|_{W^{1,p}(\Omega)} \leq C_p \|\nabla \times u\|_{L^p(\Omega)}. \tag{2.3}$$

Let $w = u - h$ in Ω. Note that $w \in L^p(\Omega; \mathbb{R}^3)$ and $\nabla \times w = 0$ in Ω. It then follows from Theorem 2.1 that there exists $P \in W^{1,p}(\Omega)$ such that $w = \nabla P$ in Ω.

We now observe that

$$\mathrm{div}(A \nabla P) = \mathrm{div}(Au) - \mathrm{div}(Ah) \in L^p(\Omega) \tag{2.4}$$

and

$$n \times \nabla P = n \times u - n \times h \in W^{1-\frac{1}{p},p}(\partial\Omega), \tag{2.5}$$

where we have used the fact that Ω is $C^{1,1}$ and

$$\|n \times h\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C \|h\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \leq C \|h\|_{W^{1,p}(\Omega)} \leq C \|\nabla \times u\|_{L^p(\Omega)}. \tag{2.6}$$

Finally, we note that if $\int_{\partial\Omega} P = 0$,

$$\|P\|_{W^{2-\frac{1}{p},p}(\partial\Omega)} \le C\, \|\nabla_{\tan} P\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \le C\, \|n \times \nabla P\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}.$$

It follows from the $W^{2,p}$ estimates for elliptic equations in $C^{1,1}$ domains (see e.g. [6]) that

$$\begin{aligned} \|u\|_{W^{1,p}(\Omega)} &\le \|h\|_{W^{1,p}(\Omega)} + \|\nabla P\|_{W^{1,p}(\Omega)} \\ &\le C\left\{ \|\nabla \times u\|_{L^p(\Omega)} + \|\mathrm{div}(Au)\|_{L^p(\Omega)} + \|n \times u\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \right\}. \end{aligned} \tag{2.7}$$

This completes the proof. $\square$

Remark 2.4. Assume that Ω is a bounded, simply-connected, $C^{1,1}$ domain in $\mathbb{R}^3$ with connected boundary and that B is a positive-definite constant matrix. Let u_ε be a solution of (1.7). It follows from (2.2) that

$$\begin{aligned} \|\nabla u_\varepsilon\|_{W^{1,p}(\Omega)} &\le C_p \left\{ \|\nabla \times u_\varepsilon\|_{L^p(\Omega)} + \|\mathrm{div}(Bu_\varepsilon)\|_{L^p(\Omega)} + \|n \times u_\varepsilon\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \right\} \\ &= C_p \left\{ \|\nabla \times u_\varepsilon\|_{L^p(\Omega)} + \|\mathrm{div}(F)\|_{L^p(\Omega)} + \|n \times u_\varepsilon\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \right\}. \end{aligned}$$

This, together with (1.10), gives (1.15).

3 Uniform estimates for scalar elliptic equations with periodic coefficients

In this section we establish the $W^{1,p}$ and Lipschitz estimates for solutions of the elliptic equation (1.14). These estimates will be used in the proof of Theorems 1.1 and 1.2.

Let $A = A(y) = (a_{ij}(y))$ be a $d \times d$ matrix in $\mathbb{R}^d$, $d \ge 2$. We say $A \in \Lambda(\mu, \lambda, \tau)$ for some $\mu > 0$, $\tau \in (0, 1]$, and $\lambda \ge 0$, if A satisfies the ellipticity condition,

$$\mu|\xi|^2 \le a_{ij}(y)\xi_i\xi_j \le \frac{1}{\mu}|\xi|^2 \quad \text{for any } y \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d, \tag{3.1}$$

the periodicity condition,

$$A(y + z) = A(y) \quad \text{for any } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d, \tag{3.2}$$

and the smoothness condition,

$$|A(x) - A(y)| \le \lambda|x - y|^\tau \text{ for any } x, y \in \mathbb{R}^d. \tag{3.3}$$

We start out with the $W^{1,p}$ estimate for solutions of the Dirichlet problem.

Theorem 3.1. *Let $1 < p < \infty$ and Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$, $d \ge 2$. Suppose that $A \in \Lambda(\mu, \lambda, \tau)$. Let $w_\varepsilon \in W^{1,p}(\Omega)$ be the solution of the Dirichlet problem:*

$$\begin{cases} \mathrm{div}\left\{A(x/\varepsilon)(\nabla w_\varepsilon + g)\right\} = \mathrm{div}(F) & \text{in } \Omega, \\ w_\varepsilon = f & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $g \in L^p(\Omega;\mathbb{R}^d)$, $F \in L^p(\Omega;\mathbb{R}^d)$, *and* $f \in W^{1-\frac{1}{p},p}(\partial\Omega)$. *Then,*

$$\|w_\varepsilon\|_{W^{1,p}(\Omega)} \le C_p \left\{ \|g\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} + \|f\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \right\}, \tag{3.5}$$

where C_p *depends only on* p, μ, λ, τ, *and* Ω.

Proof. Rewrite the elliptic equation in (3.4) as

$$\operatorname{div}\{A(x/\varepsilon)\nabla w_\varepsilon\} = \operatorname{div}\big(F - A(x/\varepsilon)g\big). \tag{3.6}$$

The estimate (3.5) is a simple consequence of [2, Theorem C]. □

The next theorem establishes the Lipschitz estimate for solutions of the Dirichlet problem.

Theorem 3.2. *Suppose that* A *and* Ω *satisfy the same assumptions as in Theorem 3.1. Let* $g \in C^\gamma(\Omega;\mathbb{R}^d)$, $F \in C^\gamma(\Omega;\mathbb{R}^d)$, *and* $f \in C^{1,\gamma}(\partial\Omega)$ *for some* $0 < \gamma < \alpha$. *Let* $w_\varepsilon \in H^1(\Omega)$ *be the solution of the Dirichlet problem (3.4). Then* $\nabla w_\varepsilon \in L^\infty(\Omega)$ *and*

$$\|\nabla w_\varepsilon\|_{L^\infty(\Omega)} \le C_\gamma \left\{ \|g\|_{C^\gamma(\Omega)} + \|F\|_{C^\gamma(\Omega)} + \|f\|_{C^{1,\gamma}(\partial\Omega)} \right\}, \tag{3.7}$$

where C_γ *depends only on* γ, μ, λ, τ, *and* Ω.

Proof. We begin by choosing $h \in C^{1,\gamma}(\overline{\Omega})$ so that $h = f$ on $\partial\Omega$ and $\|h\|_{C^{1,\gamma}(\Omega)} \le C\,\|f\|_{C^{1,\gamma}(\partial\Omega)}$. By considering $w_\varepsilon - h$, we may assume that $f = 0$. Next, in view of (3.6), we may write

$$\begin{aligned} w_\varepsilon(x) &= -\int_\Omega \frac{\partial}{\partial y_i}\{G_\varepsilon(x,y)\} a_{ij}(y/\varepsilon) g_j(y)\, dy + \int_\Omega \frac{\partial}{\partial y_i}\{G_\varepsilon(x,y)\} F_i(y)\, dy \\ &= w_\varepsilon^{(1)}(x) + w_\varepsilon^{(2)}(x), \end{aligned} \tag{3.8}$$

where $F = (F_1, \dots, F_d)$ and $G_\varepsilon(x,y)$ denotes the Green function for the operator $-\operatorname{div}(A(x/\varepsilon)\nabla)$ in Ω, with pole at y. It follows from [1] that for any $x, y \in \Omega$,

$$\begin{aligned} |G_\varepsilon(x,y)| &\le C|x-y|^{2-d}, \\ |\nabla_x G_\varepsilon(x,y)| + |\nabla_y G_\varepsilon(x,y)| &\le C|x-y|^{1-d}, \\ |\nabla_x \nabla_y G_\varepsilon(x,y)| &\le C|x-y|^{-d}, \end{aligned} \tag{3.9}$$

where C depends only on μ, λ, τ, and Ω. We note that if $d = 2$, the first inequality in (3.9) should be replaced by $|G_\varepsilon(x,y)| \le C(1 + \log|x-y|)$. Using (3.9), we see that for any $x \in \Omega$,

$$\begin{aligned} |\nabla w_\varepsilon^{(2)}(x)| &= \left| \int_\Omega \frac{\partial}{\partial y_i}\big\{\nabla_x G_\varepsilon(x,y)\big\}\big\{F_i(y) - F_i(x)\big\}\, dy \right| \\ &\le C\,\|F\|_{C^\gamma(\Omega)} \int_\Omega \frac{dy}{|x-y|^{d-\gamma}} \\ &\le C\,\|F\|_{C^\gamma(\Omega)}. \end{aligned} \tag{3.10}$$

Finally, to estimate $\nabla w_\varepsilon^{(1)}$, we let $\Phi_\varepsilon(x)$ be the Dirichlet corrector for the operator $-\operatorname{div}(A(x/\varepsilon)\nabla)$ in Ω; i.e., $\Phi_\varepsilon = (\Phi_{\varepsilon,1}(x), \dots, \Phi_{\varepsilon,d}(x))$ is the function in $H^1(\Omega;\mathbb{R}^d)$ satisfying

$$\begin{cases} \operatorname{div}\big(A(x/\varepsilon)\nabla\Phi_{\varepsilon,k}\big) = 0 & \text{in } \Omega, \\ \Phi_{\varepsilon,k} = x_k & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Since $\Phi_{\varepsilon,k} - x_k = 0$ on $\partial\Omega$ and

$$-\text{div}\big\{A(x/\varepsilon)\nabla\big(\Phi_{\varepsilon,k} - x_k\big)\big\} = \frac{\partial}{\partial x_i}\big\{a_{ik}(x/\varepsilon)\big\} \quad \text{ in } \Omega,$$

we see that

$$\Phi_{\varepsilon,k}(x) - x_k = -\int_\Omega \frac{\partial}{\partial y_i}\big\{G_\varepsilon(x,y)\big\}a_{ik}(y/\varepsilon)\,dy \tag{3.12}$$

for $1 \le k \le d$. It follows that

$$\begin{aligned}
|\nabla w_\varepsilon^{(1)}(x)| &= \left|\int_\Omega \frac{\partial}{\partial y_i}\big\{\nabla_x G_\varepsilon(x,y)\big\}\big\{a_{ij}(y/\varepsilon)g_j(y) - a_{ij}(x/\varepsilon)g_j(x)\big\}\,dy\right| \\
&\le \int_\Omega |\nabla_x\nabla_y G_\varepsilon(x,y)|\,|A(y/\varepsilon)|\,|g(y)-g(x)|\,dy \\
&\quad + \left|g_j(x)\int_\Omega \frac{\partial}{\partial y_i}\big\{\nabla_x G_\varepsilon(x,y)\big\}\big\{a_{ij}(y/\varepsilon) - a_{ij}(x/\varepsilon)\big\}\,dy\right| \\
&\le C\,\|g\|_{C^\gamma(\Omega)} + |g_j(x)|\,|\nabla\{\Phi_{\varepsilon,j}(x) - x_j\}|,
\end{aligned}$$

where we have used (3.12) and the estimate $|\nabla_x\nabla_y G_\varepsilon(x,y)| \le C|x-y|^{-d}$ for the last inequality. This, together with the Lipschitz estimate $\|\nabla\Phi_\varepsilon\|_{L^\infty(\Omega)} \le C$, established in [1], yields $\|\nabla w_\varepsilon^{(1)}\|_{L^\infty(\Omega)} \le C\,\|g\|_{C^\gamma(\Omega)}$. The proof is complete. □

We now turn to the $W^{1,p}$ estimate for solutions of the Neumann problem.

Theorem 3.3. *Let $1 < p < \infty$ and Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$, $d \ge 2$. Suppose that $A \in \Lambda(\mu,\lambda,\tau)$. Let $w_\varepsilon \in W^{1,p}(\Omega)$ be a solution of the Neumann problem:*

$$\begin{cases} \text{div}\,\{A(x/\varepsilon)(\nabla w_\varepsilon + g)\} = 0 & \text{in } \Omega, \\ n\cdot A(x/\varepsilon)(\nabla w_\varepsilon + g) = f & \text{on } \partial\Omega, \end{cases} \tag{3.13}$$

where $g \in L^p(\Omega;\mathbb{R}^d)$, $f \in W^{-\frac{1}{p},p}(\partial\Omega)$ and $< f, 1 >= 0$. Then

$$\|\nabla w_\varepsilon\|_{L^p(\Omega)} \le C_p\left\{\|g\|_{L^p(\Omega)} + \|f\|_{W^{-\frac{1}{p},p}(\partial\Omega)}\right\}, \tag{3.14}$$

where C_p depends only on p, μ, λ, τ, and Ω.

Proof. This is a direct consequence of Theorem 1.1 in [8]. □

The next theorem gives the Lipschitz estimate for solutions of the Neumann problem (3.13). Note that in addition to the ellipticity and periodicity conditions, we also assume that $A^* = A$; i.e., $a_{ij}(y) = a_{ji}(y)$.

Theorem 3.4. *Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$, $d \ge 2$. Suppose that $A \in \Lambda(\mu,\lambda,\tau)$ and $A^* = A$. Let $g \in C^\gamma(\Omega;\mathbb{R}^d)$, and $f \in C^\gamma(\partial\Omega)$ with mean value zero, for some $0 < \gamma < \alpha$. Let $w_\varepsilon \in H^1(\Omega)$ be a solution of the Neumann problem (3.13). Then $\nabla u_\varepsilon \in L^\infty(\Omega)$, and*

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le C_\gamma\left\{\|g\|_{C^\gamma(\Omega)} + \|f\|_{C^\gamma(\partial\Omega)}\right\}, \tag{3.15}$$

where C_γ depends only on γ, μ, λ, τ, and Ω.

Proof. Let $v_\varepsilon \in H^1(\Omega)$ be a solution of the Neumann problem: $\text{div}\big(A(x/\varepsilon)\nabla v_\varepsilon\big) = 0$ in Ω and $n \cdot A(x/\varepsilon)\nabla v_\varepsilon = f$ on $\partial\Omega$. It follows from [8, Theorem 1.2] that

$$\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \le C\, \|f\|_{C^\gamma(\partial\Omega)},$$

where C depends only on γ, μ, λ, τ, and Ω. Thus, by considering $w_\varepsilon - v_\varepsilon$, we may assume that $f = 0$.

Let $g = (g_1, \ldots, g_d) \in C^\gamma(\Omega; \mathbb{R}^d)$ and w_ε be a solution of (3.13) with $f = 0$. Then

$$w_\varepsilon(x) = -\int_\Omega \frac{\partial}{\partial y_i}\big\{N_\varepsilon(x,y)\big\} a_{ij}(y/\varepsilon) g_j(y)\, dy + E$$

for some constant E, where $N_\varepsilon(x,y)$ denotes the Neumann function for the elliptic operator $-\text{div}(A(x/\varepsilon)\nabla)$ in Ω, with pole at y. Under the assumption that $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$, it was proved in [8] that for $d \ge 3$,

$$\begin{aligned} |N_\varepsilon(x,y) &\le C|x-y|^{2-d}, \\ |\nabla_x N_\varepsilon(x,y)| + |\nabla_y N_\varepsilon(x,y)| &\le C|x-y|^{1-d}, \\ |\nabla_x \nabla_y N_\varepsilon(x,y)| &\le C|x-y|^{-d}, \end{aligned} \tag{3.16}$$

where C depends only on μ, λ, τ, and Ω. If $d = 2$, one obtains $|N_\varepsilon(x,y)| \le C_\eta |x-y|^{-\eta}$, $|\nabla_x N_\varepsilon(x,y)| + |\nabla_y N_\varepsilon(x,y)| \le C_\eta |x-y|^{-1-\eta}$, and $|\nabla_x\nabla_y N_\varepsilon(x,y)| \le C_\eta|x-y|^{-2-\eta}$ for any $\eta > 0$ (this is not sharp, but enough for the proof of this theorem). It follows that for any $x \in \Omega$,

$$\begin{aligned} \nabla w_\varepsilon(x) = &-\int_\Omega \frac{\partial}{\partial y_i}\big\{\nabla_x N_\varepsilon(x,y)\big\}\big[a_{ij}(y/\varepsilon)g_j(y) - a_{ij}(x/\varepsilon)g_j(x)\big]\, dy \\ &\quad - a_{ij}(x/\varepsilon) g_j(x) \int_{\partial\Omega} n_i(y) \nabla_x N_\varepsilon(x,y)\, d\sigma(y) \\ = &-\int_\Omega \frac{\partial}{\partial y_i}\big\{\nabla_x N_\varepsilon(x,y)\big\}\big[g_j(y) - g_j(x)\big] a_{ij}(y/\varepsilon)\, dy \\ &\quad - g_j(x)\int_\Omega \frac{\partial}{\partial y_i}\big\{\nabla_x N_\varepsilon(x,y)\big\}\big[a_{ij}(y/\varepsilon) - a_{ij}(x/\varepsilon)\big]\, dy \\ &\quad - a_{ij}(x/\varepsilon) g_j(x) \int_{\partial\Omega} n_i(y)\nabla_x N_\varepsilon(x,y)\, d\sigma(y). \end{aligned} \tag{3.17}$$

Note that if $g_j(x) = -\delta_{jk}$, then $w_\varepsilon(x) = x_k$ is a solution of (3.13) with $f = 0$. In view of (3.17), this implies that

$$\begin{aligned} \nabla(x_k) = \delta_{jk} &\int_\Omega \frac{\partial}{\partial y_i}\big\{\nabla_x N_\varepsilon(x,y)\big\}\big[a_{ij}(y/\varepsilon) - a_{ij}(x/\varepsilon)\big]\, dy \\ &+ a_{ij}(x/\varepsilon)\delta_{jk} \int_{\partial\Omega} n_i(y) \nabla_x N_\varepsilon(x,y)\, d\sigma(y). \end{aligned} \tag{3.18}$$

By combining (3.17) and (3.18) we obtain

$$\nabla w_\varepsilon(x) + g_j(x)\nabla(x_j) = -\int_\Omega \frac{\partial}{\partial y_i}\big\{\nabla_x N_\varepsilon(x,y)\big\}\big[g_j(y) - g_j(x)\big] a_{ij}(y/\varepsilon)\, dy.$$

As a result, for any $x \in \Omega$,

$$
\begin{aligned}
|\nabla w_\varepsilon(x)| &\le C\|g\|_{L^\infty(\Omega)} + C\,\|g\|_{C^\gamma(\Omega)} \int_\Omega \frac{dy}{|x-y|^{d-\gamma}} \\
&\le C\|g\|_{C^\gamma(\Omega)},
\end{aligned}
$$

where we have used the estimate $|\nabla_x \nabla_y N_\varepsilon(x,y)| \le C|x-y|^{-d}$ (the case $d=2$ may be handled in a similar manner). This finishes the proof. $\square$

4 L^p estimates

The goal of this section is to prove Theorem 1.1. Throughout this section we will assume that Ω is a bounded, simply connected, $C^{1,\alpha}$ domain in $\mathbb{R}^3$ with connected boundary, and that $A, B \in \Lambda(\mu, \lambda, \tau)$.

Lemma 4.1. *Let $2 \le q < 3$ and $2 \le p \le p_0$, where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{3}$. Given $F \in L^q(\Omega;\mathbb{R}^3)$ and $G \in L^p(\Omega;\mathbb{R}^3)$, let u_ε be the unique solution in $V_0^2(\Omega)$ of (1.7) with $f = 0$. Suppose that $u_\varepsilon \in L^q(\Omega;\mathbb{R}^3)$. Then $\nabla \times u_\varepsilon \in L^p(\Omega;\mathbb{R}^3)$, and*

$$
\|\nabla \times u_\varepsilon\|_{L^p(\Omega)} \le C\left\{\|F\|_{L^q(\Omega)} + \|G\|_{L^p(\Omega)} + \|u_\varepsilon\|_{L^q(\Omega)}\right\}, \tag{4.1}
$$

where C depends only on q, μ, λ, τ, and Ω.

Proof. It follows from the elliptic system in (1.2) that

$$
\operatorname{div}\big(B(x/\varepsilon)u_\varepsilon - F\big) = 0 \quad \text{ in } \Omega.
$$

Since $B(x/\varepsilon)u_\varepsilon - F \in L^q(\Omega;\mathbb{R}^3)$, by Theorem 2.2, there exists $h_\varepsilon \in W^{1,q}(\Omega;\mathbb{R}^3)$ such that

$$
\nabla \times h_\varepsilon = B(x/\varepsilon)u_\varepsilon - F \quad \text{ in } \Omega, \tag{4.2}
$$

and

$$
\|h_\varepsilon\|_{W^{1,q}(\Omega)} \le C_q\,\|B(x/\varepsilon)u_\varepsilon - F\|_{L^q(\Omega)}. \tag{4.3}
$$

Thus,

$$
\nabla \times \big\{A(x/\varepsilon)\nabla \times u_\varepsilon + h_\varepsilon - G\big\} = 0 \quad \text{ in } \Omega.
$$

Since Ω is simply connected, there exists $P_\varepsilon \in W^{1,2}(\Omega)$ such that

$$
A(x/\varepsilon)\nabla \times u_\varepsilon + h_\varepsilon - G = \nabla P_\varepsilon \quad \text{ in } \Omega. \tag{4.4}
$$

It follows that

$$
A^{-1}(x/\varepsilon)\nabla P_\varepsilon = \nabla \times u_\varepsilon + A^{-1}(x/\varepsilon)\big(h_\varepsilon - G\big) \quad \text{ in } \Omega. \tag{4.5}
$$

Thus, $P_\varepsilon \in W^{1,2}(\Omega)$ is the solution of

$$
\begin{cases}
\operatorname{div}\big\{A^{-1}(x/\varepsilon)\big(\nabla P_\varepsilon - h_\varepsilon + G\big)\big\} = 0 & \text{ in } \Omega, \\
n \cdot A^{-1}(x/\varepsilon)\big(\nabla P_\varepsilon - h_\varepsilon + G\big) = 0 & \text{ on } \partial\Omega,
\end{cases} \tag{4.6}
$$

where we have used the fact that

$$n \cdot (\nabla \times u_\varepsilon) = -\text{Div}(n \times u_\varepsilon) = 0 \quad \text{on } \partial\Omega.$$

In view of Theorem 3.3, we obtain

$$\|\nabla P_\varepsilon\|_{L^p(\Omega)} \le C \left\{ \|h_\varepsilon\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)} \right\},$$

where C depends only on p, μ, λ, τ, and Ω. This, together with (4.5) and the estimate (4.3), gives

$$\begin{aligned} \|\nabla \times u_\varepsilon\|_{L^p(\Omega)} &\le C \left\{ \|h_\varepsilon\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)} \right\} \\ &\le C \left\{ \|h_\varepsilon\|_{W^{1,q}(\Omega)} + \|G\|_{L^p(\Omega)} \right\} \\ &\le C \left\{ \|F\|_{L^q(\Omega)} + \|G\|_{L^p(\Omega)} + \|u_\varepsilon\|_{L^q(\Omega)} \right\}, \end{aligned} \tag{4.7}$$

where we also used the Sobolev imbedding for the second inequality. □

Remark 4.2. Let $F \in L^q(\Omega; \mathbb{R}^3)$, $G \in C^\gamma(\Omega; \mathbb{R}^3)$, and $f \in C^\gamma(\partial\Omega; \mathbb{R}^3)$ with $n \cdot f = 0$ on $\partial\Omega$ and $\text{Div}(f) \in C^\gamma(\partial\Omega)$, where $3 < q < \infty$ and $\gamma = 1 - \frac{3}{q} < \alpha$. Let u_ε be the solution in $V^2(\Omega)$ of (1.7). Suppose that A is symmetric. Then

$$\|\nabla \times u_\varepsilon\|_{L^\infty(\Omega)} \le C \left\{ \|F\|_{L^q(\Omega)} + \|G\|_{C^\gamma(\Omega)} + \|\text{Div}(f)\|_{C^\gamma(\partial\Omega)} + \|u_\varepsilon\|_{L^q(\Omega)} \right\}, \tag{4.8}$$

where C depends only q, μ, λ, τ, and Ω. To see this, we let $h_\varepsilon \in W^{1,q}(\Omega; \mathbb{R}^3)$ and $P_\varepsilon \in W^{1,2}(\Omega)$ be the same functions as in the proof of Lemma 4.1. It follows from (4.4), (4.6) and Theorem 3.4 that

$$\begin{aligned} \|\nabla \times u_\varepsilon\|_{L^\infty(\Omega)} &\le C \left\{ \|\nabla P_\varepsilon\|_{L^\infty(\Omega)} + \|h_\varepsilon\|_{L^\infty(\Omega)} + \|G\|_{L^\infty(\Omega)} \right\} \\ &\le C \left\{ \|h_\varepsilon\|_{C^\gamma(\Omega)} + \|G\|_{C^\gamma(\Omega)} + \|\text{Div}(f)\|_{C^\gamma(\partial\Omega)} \right\} \\ &\le C \left\{ \|h_\varepsilon\|_{W^{1,q}(\Omega)} + \|G\|_{C^\gamma(\Omega)} + \|\text{Div}(f)\|_{C^\gamma(\partial\Omega)} \right\} \\ &\le C \left\{ \|F\|_{L^q(\Omega)} + \|G\|_{C^\gamma(\Omega)} + \|\text{Div}(f)\|_{C^\gamma(\partial\Omega)} + \|u_\varepsilon\|_{L^q(\Omega)} \right\}, \end{aligned}$$

where we have used the Sobolev imbedding for the third inequality and (4.3) for the last.

Next we reverse the roles of u_ε and $\nabla \times u_\varepsilon$ in the estimate (4.1).

Lemma 4.3. *Let $2 \le q < 3$ and $2 \le p \le p_0$, where $\frac{1}{p_0} = \frac{1}{q} - \frac{1}{3}$. Given $F \in L^p(\Omega; \mathbb{R}^3)$ and $G \in L^q(\Omega; \mathbb{R}^3)$, let u_ε be the unique solution in $V_0^2(\Omega)$ of (1.7) with $f = 0$. Suppose that $\nabla \times u_\varepsilon \in L^q(\Omega; \mathbb{R}^3)$. Then $u_\varepsilon \in L^p(\Omega; \mathbb{R}^3)$ and*

$$\|u_\varepsilon\|_{L^p(\Omega)} \le C \left\{ \|F\|_{L^p(\Omega)} + \|G\|_{L^q(\Omega)} + \|\nabla \times u_\varepsilon\|_{L^q(\Omega)} \right\}, \tag{4.9}$$

where C depends only on q, μ, λ, τ, and Ω.

Proof. Let

$$v_\varepsilon = A(x/\varepsilon)\nabla \times u_\varepsilon - G \quad \text{in } \Omega. \tag{4.10}$$

Then $\nabla \times u_\varepsilon = A^{-1}(x/\varepsilon)\big(v_\varepsilon + G\big)$ in Ω. It follows that

$$\text{div}\big(A^{-1}(x/\varepsilon)(v_\varepsilon + G)\big) = 0 \quad \text{in } \Omega.$$

By Theorem 2.2 there exists $h_\varepsilon \in W^{1,q}(\Omega;\mathbb{R}^3)$ such that

$$A^{-1}(x/\varepsilon)(v_\varepsilon + G) = \nabla \times h_\varepsilon \quad \text{ in } \Omega \tag{4.11}$$

and

$$\|h_\varepsilon\|_{W^{1,q}(\Omega)} \le C\left\{\|v_\varepsilon\|_{L^q(\Omega)} + \|G\|_{L^q(\Omega)}\right\} \le C\left\{\|\nabla\times u_\varepsilon\|_{L^q(\Omega)} + \|G\|_{L^q(\Omega)}\right\}. \tag{4.12}$$

Note that by the elliptic system (1.2), $\nabla \times v_\varepsilon = -B(x/\varepsilon)u_\varepsilon + F$ in Ω. Thus,

$$u_\varepsilon = B^{-1}(x/\varepsilon)F - B^{-1}(x/\varepsilon)\nabla\times v_\varepsilon \quad \text{ in } \Omega. \tag{4.13}$$

Since Ω is simply connected and $\nabla\times h_\varepsilon = \nabla\times u_\varepsilon$ in Ω, there exists $Q_\varepsilon \in W^{1,2}(\Omega)$ such that $\nabla Q_\varepsilon = u_\varepsilon - h_\varepsilon$ in Ω. Thus,

$$B(x/\varepsilon)\big(\nabla Q_\varepsilon + h_\varepsilon\big) = B(x/\varepsilon)u_\varepsilon = F - \nabla\times v_\varepsilon \quad \text{ in } \Omega.$$

It follows that

$$\operatorname{div}\big\{B(x/\varepsilon)\big(\nabla Q_\varepsilon + h_\varepsilon\big)\big\} = \operatorname{div}\big(F\big) \quad \text{ in } \Omega. \tag{4.14}$$

In view of Theorem 3.1 we obtain

$$\|\nabla Q_\varepsilon\|_{L^p(\Omega)} \le C\left\{\|h_\varepsilon\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} + \|Q_\varepsilon\|_{W^{1-\frac{1}{p},p}(\partial\Omega)}\right\}, \tag{4.15}$$

where C depends only on p, μ, λ, τ, and Ω.

Finally, we note that

$$n\times\nabla Q_\varepsilon = n\times u_\varepsilon - n\times h_\varepsilon = -n\times h_\varepsilon \quad \text{ on } \partial\Omega. \tag{4.16}$$

By subtracting a constant we may assume that $\int_{\partial\Omega} Q_\varepsilon \, d\sigma = 0$. Since $|\nabla_{\tan} Q_\varepsilon| = |n\times\nabla Q_\varepsilon|$ on $\partial\Omega$, we see that

$$\begin{aligned}
\|Q_\varepsilon\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} &\le C\,\|n\times\nabla Q_\varepsilon\|_{L^{q_1}(\partial\Omega)} = C\,\|h_\varepsilon\|_{L^{q_1}(\partial\Omega)}\\
&\le C\,\|h_\varepsilon\|_{W^{1-\frac{1}{q},q}(\partial\Omega)} \le C\,\|h_\varepsilon\|_{W^{1,q}(\Omega)}\\
&\le C\left\{\|\nabla\times u_\varepsilon\|_{L^q(\Omega)} + \|G\|_{L^q(\Omega)}\right\},
\end{aligned}$$

where $q_1 = 2p/3$ and we have used the Sobolev imbedding on $\partial\Omega$ as well as the estimate (4.12). This, together with (4.15) and (4.12), gives

$$\begin{aligned}
\|u_\varepsilon\|_{L^p(\Omega)} &\le \|\nabla Q_\varepsilon\|_{L^p(\Omega)} + \|h_\varepsilon\|_{L^p(\Omega)}\\
&\le C\left\{\|F\|_{L^p(\Omega)} + \|G\|_{L^q(\Omega)} + \|\nabla\times u_\varepsilon\|_{L^q(\Omega)}\right\},
\end{aligned}$$

and completes the proof. $\square$

Remark 4.4. Let $F \in C^\gamma(\Omega;\mathbb{R}^3)$, $G \in L^q(\Omega;\mathbb{R}^3)$, and $f \in C^\gamma(\partial\Omega;\mathbb{R}^3)$ with $n\cdot f = 0$ on $\partial\Omega$, where $3 < q < \infty$ and $\gamma = 1 - \frac{3}{q} < \alpha$. Let u_ε be the solution in $V^2(\Omega)$ of (1.7). Then

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le C\left\{\|F\|_{C^\gamma(\Omega)} + \|G\|_{L^q(\Omega)} + \|f\|_{C^\gamma(\partial\Omega)} + \|\nabla\times u_\varepsilon\|_{L^q(\Omega)}\right\}, \tag{4.17}$$

where C depends only on q, μ, λ, τ, and Ω. To see this, we let $h_\varepsilon \in W^{1,q}(\Omega;\mathbb{R}^3)$ and $Q_\varepsilon \in W^{1,2}(\Omega)$ be the same functions as in the proof of Lemma 4.3. It follows from Theorem 3.2 that

$$\begin{aligned}
\|u_\varepsilon\|_{L^\infty(\Omega)} &\le \|\nabla Q_\varepsilon\|_{L^\infty(\Omega)} + \|h_\varepsilon\|_{L^\infty(\Omega)} \\
&\le C\left\{\|F\|_{C^\gamma(\Omega)} + \|h_\varepsilon\|_{C^\gamma(\Omega)} + \|\nabla_{\tan} Q_\varepsilon\|_{C^\gamma(\partial\Omega)}\right\}. \\
&\le C\left\{\|F\|_{C^\gamma(\Omega)} + \|h\|_{W^{1,q}(\Omega)} + \|\nabla_{\tan} Q_\varepsilon\|_{C^\gamma(\partial\Omega)}\right\} \\
&\le C\left\{\|F\|_{C^\gamma(\Omega)} + \|G\|_{L^q(\Omega)} + \|\nabla\times u_\varepsilon\|_{L^q(\Omega)} + \|\nabla_{\tan} Q_\varepsilon\|_{C^\gamma(\partial\Omega)}\right\},
\end{aligned} \tag{4.18}$$

where we have used the Sobolev embedding for the third inequality and (4.12) for the fourth. This, together with the estimate

$$\begin{aligned}
\|\nabla_{\tan} Q_\varepsilon\|_{C^\gamma(\partial\Omega)} &\le \|n\times\nabla Q_\varepsilon\|_{C^\gamma(\partial\Omega)} \\
&\le \|n\times u_\varepsilon\|_{C^\gamma(\partial\Omega)} + \|n\times h_\varepsilon\|_{C^\gamma(\partial\Omega)} \\
&\le \|f\|_{C^\gamma(\partial\Omega)} + C\,\|h_\varepsilon\|_{W^{1,q}(\Omega)} \\
&\le \|f\|_{C^\gamma(\partial\Omega)} + C\left\{\|\nabla\times u_\varepsilon\|_{L^q(\Omega)} + \|G\|_{L^q(\Omega)}\right\},
\end{aligned}$$

gives (4.17).

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Given $f \in L^p(\partial\Omega;\mathbb{R}^3)$ with $n\cdot f = 0$ on $\partial\Omega$ and $\mathrm{Div}(f) \in W^{-\frac{1}{p},p}(\partial\Omega)$, it follows from [11, Theorem 11.6] that there exists $u \in V^p(\Omega)$ such that $n\times u = f$ on $\partial\Omega$ and

$$\|u\|_{L^p(\Omega)} + \|\nabla\times u\|_{L^p(\Omega)} \le C_p\left\{\|f\|_{L^p(\partial\Omega)} + \|\mathrm{Div}(f)\|_{W^{-\frac{1}{p},p}(\partial\Omega)}\right\}.$$

Consequently, by considering $u_\varepsilon - u$ in Ω, we may assume that $f = 0$.

We first consider the case $p > 2$. The uniqueness follows from the uniqueness in the case $p = 2$. Let $F \in L^p(\Omega;\mathbb{R}^3)$, $G \in L^p(\Omega;\mathbb{R}^3)$, and u_ε be the unique solution of (1.2) in $V_0^2(\Omega)$. To establish the L^p estimate (1.10), we further assume that $2 < p \le 6$. Since $\|u_\varepsilon\|_{L^2(\Omega)} + \|\nabla\times u_\varepsilon\|_{L^2(\Omega)} \le C\left\{\|F\|_{L^2(\Omega)} + \|G\|_{L^2(\Omega)}\right\}$, it follows from (4.1) that

$$\begin{aligned}
\|\nabla\times u_\varepsilon\|_{L^p(\Omega)} &\le C\left\{\|F\|_{L^2(\Omega)} + \|G\|_{L^p(\Omega)} + \|u_\varepsilon\|_{L^2(\Omega)}\right\} \\
&\le C\left\{\|F\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)}\right\}.
\end{aligned}$$

Similarly, by the estimate (4.9),

$$\begin{aligned}
\|u_\varepsilon\|_{L^p(\Omega)} &\le C\left\{\|F\|_{L^p(\Omega)} + \|G\|_{L^2(\Omega)} + \|\nabla\times u_\varepsilon\|_{L^2(\Omega)}\right\} \\
&\le C\left\{\|F\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)}\right\}.
\end{aligned}$$

Suppose now that $p > 6$. Let $\frac{1}{q} = \frac{1}{p} + \frac{1}{3}$. Then $2 < q < 3$ and we have proved that

$$\|u_\varepsilon\|_{L^q(\Omega)} + \|\nabla\times u_\varepsilon\|_{L^q(\Omega)} \le C\left\{\|F\|_{L^q(\Omega)} + \|G\|_{L^q(\Omega)}\right\}.$$

As before, we may use estimates (4.1) and (4.9) to obtain

$$\begin{aligned}\|\nabla \times u_\varepsilon\|_{L^p(\Omega)} &\le C\left\{\|F\|_{L^q(\Omega)} + \|G\|_{L^p(\Omega)} + \|u_\varepsilon\|_{L^q(\Omega)}\right\}\\ &\le C\left\{\|F\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)}\right\},\end{aligned}$$

and

$$\begin{aligned}\|u_\varepsilon\|_{L^p(\Omega)} &\le C\left\{\|F\|_{L^p(\Omega)} + \|G\|_{L^q(\Omega)} + \|\nabla \times u_\varepsilon\|_{L^q(\Omega)}\right\}\\ &\le C\left\{\|F\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)}\right\}.\end{aligned}$$

Finally, we handle the case $1 < p < 2$ by a duality argument. Let $F, G \in C_0^\infty(\Omega;\mathbb{R}^3)$ and u_ε be the solution in $V_0^2(\Omega)$ of (1.2). Let F_1, $G_1 \in C_0^\infty(\Omega;\mathbb{R}^3)$ and v_ε be the solution in $V_0^2(\Omega)$ of

$$\begin{cases}\nabla \times \big(A^*(x/\varepsilon)\nabla \times v_\varepsilon\big) + B^*(x/\varepsilon)v_\varepsilon = F_1 + \nabla \times G_1 & \text{in } \Omega,\\ n \times v_\varepsilon = 0 & \text{on } \partial\Omega,\end{cases}$$

where A^* and B^* are the adjoints of A and B, respectively. Since A^* and B^* satisfy the same conditions as A and B, we see that

$$\|v_\varepsilon\|_{L^{p'}(\Omega)} + \|\nabla \times v_\varepsilon\|_{L^{p'}(\Omega)} \le C\left\{\|F_1\|_{L^{p'}(\Omega)} + \|G_1\|_{L^{p'}(\Omega)}\right\}. \tag{4.19}$$

Note that

$$\begin{aligned}&\int_\Omega F_1 \cdot u_\varepsilon\, dx + \int_\Omega G_1 \cdot \nabla \times u_\varepsilon\, dx\\ &= \int_\Omega A(x/\varepsilon)\nabla \times u_\varepsilon \cdot \nabla \times v_\varepsilon\, dx + \int_\Omega B(x/\varepsilon)u_\varepsilon \cdot v_\varepsilon\, dx\\ &= \int_\Omega F \cdot v_\varepsilon\, dx + \int_\Omega G \cdot \nabla \times v_\varepsilon\, dx.\end{aligned} \tag{4.20}$$

This, together with (4.19), yields

$$\|u_\varepsilon\|_{L^p(\Omega)} + \|\nabla \times u_\varepsilon\|_{L^p(\Omega)} \le C\left\{\|F\|_{L^p(\Omega)} + \|G\|_{L^p(\Omega)}\right\}, \tag{4.21}$$

by duality. With the estimate (4.21) at our disposal, the existence of solutions in $V^p(\Omega)$ as well as the estimate (1.10) for arbitrary data $F, G \in L^p(\Omega;\mathbb{R}^3)$ follows readily by a density argument. Observe that the duality relation (4.20) holds as long as $u_\varepsilon \in V^p(\Omega)$ and $v_\varepsilon \in V^{p'}(\Omega)$ are solutions of (1.2) and its adjoint system, respectively. The uniqueness for $p < 2$ also follows from (4.20) and (4.19) by duality. This completes the proof of Theorem 1.1. □

5 Proof of Theorem 1.2

Choose $q > 3$ such that $\gamma = 1 - \frac{3}{q}$. It follows from estimates (4.8) and (4.17) that

$$\begin{aligned}&\|u_\varepsilon\|_{L^\infty(\Omega)} + \|\nabla \times u_\varepsilon\|_{L^\infty(\Omega)}\\ &\qquad \le C\big\{\|F\|_{C^\gamma(\Omega)} + \|G\|_{C^\gamma(\Omega)} + \|f\|_{C^\gamma(\partial\Omega)} + \|\mathrm{Div}(f)\|_{C^\gamma(\partial\Omega)}\\ &\qquad\qquad + \|u_\varepsilon\|_{L^q(\Omega)} + \|\nabla \times u_\varepsilon\|_{L^q(\Omega)}\big\}.\end{aligned}$$

This, together with the L^q estimate of u_ε and $\nabla \times u_\varepsilon$ in Theorem 1.1, gives (1.11).

References

[1] M. Avellaneda and F. Lin, *Compactness methods in the theory of homogenization*, Comm. Pure Appl. Math. **40** (1987), 803–847.

[2] ———, *L^p bounds on singular integrals in homogenization*, Comm. Pure Appl. Math. **44** (1991), 897–910.

[3] A. Bensoussan, J.-L. Lions, and G.C. Papanicolaou, *Asymptotic Analysis for Periodic Structures*, AMS Chelsea Publishing, Providence, Rhode Island, 2011.

[4] M. Costabel, *A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains*, Math. Methods Appl. Sci. **12** (1990), 365–368.

[5] E. Fabes, O. Mendez, and M. Mitrea, *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains*, J. Funct. Anal. **159** (1998), 323–368.

[6] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.

[7] V. Girault and P. Ravirart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, 1985.

[8] C. Kenig, F. Lin, and Z. Shen, *Homogenization of elliptic systems with Neumann boundary conditions*, Preprint, arXiv:1010.6114 (2010).

[9] C. Kenig and Z. Shen, *Homogenization of elliptic boundary value problems in Lipschitz domains*, Math. Ann. **350** (2011), 867–917.

[10] ———, *Layer potential methods for elliptic homogenization problems*, Comm. Pure Appl. Math. **64** (2011), 1–44.

[11] D. Mitrea, M. Mitrea, and J. Pipher, *Vector potential theory on nonsmooth domains in $\mathbb{R}^3$ and applications to electromagnetic scattering*, J. Fourier Anal. Appl. **3** (1997), 131–192.

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